DYNAMICS OF THE PIANOFORTE STRING AND THE HAMMER--PART I (HARD HAMMER)

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ABSTRACT. Dynamics of the damped pianoforte string and the hard hammer has been worked out by the help of well-known operational method. The main idea upon which the dynamics is built up is that the string behaves like *a loaded string* so long as the hammer is in contact with it. The different cases have been worked out: (1) The hammer strikes very near the end. The expressions for the displacement and the pressure when the string is free from damping are found to be the same as those given by Kaufmann. But this method does not require to assume like Kaufmann's that the shorter segment vibrates as a rigid rod during impact which however goes *against* experimental observations. For massive hammer, expressions reduce to those that were obtained independently in a different paper. (2) The general expressions for the displacement and pressure have been also obtained, for the string struck at mid-point. Kaufmann tried this with the help of St. Venant's 'variation of integration constant.' But the method fails to include the damping of the string and also fails to give general expressions. (3) The general expressions for the same are also given for the semi-infinite string struck at different points from the finite end. Das also tried this with the help of the method adopted by Kaufmann but arrived at some algebraical difficulties.

INTRODUCTION.

The dynamics of the pianoforte string and hammer was studied by a number of workers. Kaufmann[1] studied the case where the hard hammer strikes very near the end, and at mid-point. Next S. Bhargava and R. N. Ghosh[2] and afterwards the latter writer alone studied the case of an elastic hammer striking near the end of the string. P. Das[3] studied the case of the hard and elastic hammer, striking at different points of a semi-infinite string, and pointed out a certain discrepancy in Bhargava-Ghosh's analysis. Raman and Banerjee[4] tried to apply Rayleigh's theory of loaded string, to the case where the string is struck by a hard hammer. The expressions for the displacement of the loaded point of the string and the pressure exerted by the load, come out in the form of an infinite series. Such a solution of course *assumes that the string and the load constitute a stable system and so the string remains permanently loaded*, but in the case of the struck string, it is temporarily loaded, the duration of loading being equal to the duration of impact. Recently R. N. Ghosh[5] using the same operational method as was used by Jeffreys[6] to solve the problem of the vibration of a string permanently loaded at the centre worked out the case where the load is attached at any point of a finite string and arrived at the expression similar to that of Raman and Banerjee. Iynger,[7] in a paper, made an attempt to test the convergency of the infinite series obtained by Raman and Banerjee. But these workers, as is evident from their analysis, do not differentiate the dynamics of the permanently loaded string from the dynamics of the pianoforte string and the hammer. Kar[8] pointed out that while applying the analysis for the loaded string, to obtain the solution of the vibration of the pianoforte, so long as the hammer is in contact, the idea of series should be abandoned.

None of the previous theories could explain quantitatively some of the experimental facts, such as the dependence of the duration of impact on the velocity of impact when a felt hammer impinges over a string : this was observed by Kaufmann, George,[9] Weak[10] and the author.[11]

In a series of papers the writer alone,[12] and also conjointly with Kar,[13] proceeding in the same line as was pointed out by one of the authors, extended the Rayleigh's theory of loaded string to the case of the pianoforte string struck by hard, elastic and felt hammer. From the general solution so obtained, it is found that Helmholtz's, Kaufmann's, Delemer's, Das's and other theories come as special cases. This is also able to explain the dependence of the duration of contact on the impinging velocity of the felt hammer.

All the above theories fail to represent the complete dynamics of the pianoforte string, *struck at any point of a finite string* and for any mass ratio of the hammer and the string.

In this paper which will appear in different parts, the author will study the complete dynamics, following operational method due to G. Boole and afterwards developed by O. Heaviside.[14] These operational methods to some extent are equivalent. Only difference is, that, Boole's method leaves the integration constants to be determined separately, and Heaviside's method does not require it. The former method was adopted previously by the author to solve the dynamics of the longitudinal impact of a bar and the elastic load, and the latter method was applied by H. Jeffreys to obtain the expression for the wave propagation in strings permanently loaded at the mid-point, and in strings with continuous and concentrated loads permanently attached to them. We shall also study the criteria which mathematically differentiate the dynamics of the loaded string from that of the pianoforte string and hammer.

EXPLANATION OF THE SYMBOLS USED

l = Length of the string $= a + b$;

a = Shorter segment of the string ;

b = Longer segment of the string ;

t = Variable time ;

x = Variable, measured along the length of the string (the string is fixed at $x = 0$ and $x = l$) ;

$x_1 = x - a$;

y = Displacement at any point of the string at a given time ;

y_a = Displacement of the struck point, $x = a$;

y_1 = Displacement at any point, $x < a$;

y_2 = Displacement at any point, $x > a$;

K = Coefficient of damping ;

ρ = Linear density of the string ;

$M_1 = \rho a$ = Mass of the shorter segment of the string ;

$M_2 = \rho b$ = Mass of the longer segment of string ;

$M = \rho l$ = Mass of the string $= M_1 + M_2$;

m = Mass of the hammer ;

c = Velocity of the transverse wave motion, along the string ;

T = Tension along the string $= c^2 \rho$;

$\Theta = \frac{2l}{c}$ = Period of the free vibration of the string ;

$\theta_1 = \frac{2a}{c} = \Theta \frac{a}{l}$

$\theta_2 = \frac{2b}{c} = \Theta \frac{b}{l}$

$t_n = t - n\theta_1$;

v_0 = Velocity of impact ;

$J = m v_0$;

u = The compresson of the hammer ;

$z = y_a + u$ = Displacement of the hammer ;

E = Elastic constant for the material of the hammer ;

P = Pressure exerted by the hammer ;

D = Operator $\frac{d}{dt}$;

$D_1 = D + \frac{K}{2}$.

The equation of motion of the damped string is

$$\frac{d^2y}{dt^2} + K\frac{dy}{dt} = c^2\frac{d^2y}{dx^2}\,. \tag{1}$$

Now we put the usual notation D for $\frac{d}{dt}$ and consider the coefficient of damping to be small enough so that its higher power than the first may be regarded as negligible, we get from (1),

$$\frac{d^2y}{dx^2} = \frac{D_k^2}{c^2}\,y\,. \qquad \dots \quad (1.1)$$

The hammer strikes at $x=a$, and if y_a be the displacement of the struck point, then, we get, proceeding in the usual manner,

$$y_1 = y_a\frac{\sinh D_k x/c}{\sinh D_k a/c}\,, \qquad \dots \quad (1.2)$$

$$y_2 = y_a\frac{\sinh D_k(l-x)/c}{\sinh D_k b/c}\,, \qquad \dots \quad (1.3)$$

The hard hammer striking the string is supposed to behave like a load attached to the string at $x=a$. Initially the string is straight, an impulse J is given to the load. The subsequent motion of the load is given by the equation

$$m\frac{d^2y_a}{dt^2} = T\,\Delta\left(\frac{\partial y}{\partial x}\right)_{x=a} \qquad \dots \quad (2)$$

where $\Delta\left(\frac{\partial y}{\partial x}\right)$ denotes the alteration in the value of $\left(\frac{\partial y}{\partial x}\right)$ incurred in crossing the point $x=a$ in the positive direction.

Now substituting the value for $\Delta\left(\frac{\partial y}{\partial x}\right)_{x=a}$ as obtained from eqs. (1.2) and (1.3), in eq. (2), and imposing the boundary conditions we have

$$mD^2y_a = -T\frac{D_k}{c}\left(\coth\frac{D_k a}{c} + \coth\frac{D_k b}{c}\right)y_a + JD, \qquad \dots \quad (2.1)$$

whence we have

$$y_a = \frac{1}{F(D)}\cdot v_0, \qquad \dots \quad (3)$$

where $F(D) = \left[D + \frac{T}{mc}\left(1+\frac{K}{2D}\right)\left(\coth\frac{D_k a}{c} + \coth\frac{D_k b}{c}\right)\right].$ $\qquad$ (4)

SEMI-INFINITE STRING: HAMMER STRIKES NEAR THE END.

When b is too large compared with a, that is, the hammer strikes at a distance a of a semi-infinite string, $F(D)$ becomes, as $Lt,\ b \to \infty$, $\coth \frac{Db}{c} = 1$

$$F(D) \equiv D + \frac{T}{mc}\left(1 + \frac{K}{2D}\right)\left(\coth \frac{D_k a}{c} + 1\right). \qquad \dots \quad (4.1)$$

Now $\coth \frac{D_k a}{c}$ can be expanded in a series thus (*vide* Appendix, *i*)

$$\operatorname{Coth} \frac{D_k a}{c} = \frac{c}{D_k a}\left\{1 + \frac{B_1}{2!}\left(\frac{2D_k a}{c}\right)^2 - \frac{B_3}{4!}\left(\frac{2D_k a}{c}\right)^4 + \dots\right\} \qquad (4.2)$$

where B_1, B_3, B_5 etc. are Bernoullian numbers.

(*i*) When the hammer strikes very near the end, $\frac{a}{c}$ becomes very small, so that in the expansion of $\coth \frac{D_k a}{c}$, we can retain up to the second term, so eq. (4.1) becomes

$$F(D) = D + \frac{T}{mc}\left(1 + \frac{K}{2D}\right)\left(\frac{c}{D_k a} + \frac{D_k a}{3c} + 1\right)$$

$$= \frac{1}{D}\left[D^2\left(1 + \frac{\rho a}{3m}\right) + \frac{T}{mc}\left(1 + \frac{K}{3}\frac{a}{c}\right)D + \frac{T}{ma}\left(1 + \frac{K}{2}\frac{a}{c}\right)\right]. \qquad \dots \quad (5)$$

From eq. (3)

$y_a = \frac{m}{m_0} \cdot \frac{D}{(D-q)(D-p)} \cdot v_0$, as $\frac{m}{m_0}$ is approximately equal to unity (*vide* eq. 6.4) ; so we have,

$$y_a = \frac{v_0}{q-p}\left(e^{qt} - e^{pt}\right), \qquad \dots \quad (6)$$

where q and p are the roots of the equation

$$D^2 + \frac{T}{m_0 c}\left(1 + \frac{Ka}{3c}\right)D + \frac{T}{m_0 a}\left(1 + \frac{Ka}{2c}\right) = 0, \qquad \dots \quad (6.1)$$

and are given by

$$q = -\mu + i\nu$$
$$p = -\mu - i\nu \qquad \dots \quad (6.2)$$

where,
$$\mu = \frac{T}{2m_0c}\left(1+\frac{Ka}{3c}\right),$$

$$\nu = \left[\frac{T}{m_0a}\left(1+\frac{Ka}{2c}\right) - \frac{T^2}{4m_0^2c^2}\left(1+\frac{2Ka}{3c}\right)\right]^{\frac{1}{2}}, \qquad \ldots (6.3)$$

and
$$m_0 \text{ stands for } m+\frac{\rho a}{3}. \qquad \ldots (6.4)$$

So we have from eqs. (6) and (6.2)

$$y_a = \frac{v_0}{\nu}e^{-\mu t}\sin \nu t \qquad \ldots (7)$$

When the motion of the string is free from damping we have $K=0$ in eq. (6.3) and the expression (7) becomes identically equal to what was obtained by Koufmann, where we must read

$$\mu = \frac{T}{2m_0c} \quad \text{and} \quad \nu = \sqrt{\frac{T}{m_0a} - \frac{T^2}{4m_0^2c^2}}. \qquad \ldots (8)$$

Unlike the present method, it is not possible to include the effect of damping in the analysis adopted by Kaufmann. Further, Kaufmann *assumed* that the shorter segment during impact behaves like a *rigid rod* and used $m+\frac{\rho a}{3}(=m_0)$ for the mass of the hammer m, which according to him is the *effective mass* of the hammer.

A similar assumption was also made by Das, in his paper (*vide* eq. 26 of his paper) and by Bhargava-Ghosh (*vide* eq. 12 of their paper). But the author's analysis is *free* from such assumptions (*vide* eq. 6.4).

In this case the pressure exerted by the hammer is given by $P = m\ddot{y}_a$, which by the help of eq. (7) becomes

$$P = m\frac{v_0}{\nu}(\mu^2+\nu^2)e^{-\mu t}\sin\left(\nu t - \tan^{-1}\frac{3\mu\nu}{\mu^2-\nu^2}\right) \qquad \ldots (9)$$

where μ and ν are known from eq. (6.3) or eq. (8).

(*ii*) As the magnitude of $c >> a$ we have $\frac{T}{mc} << \frac{T}{ma}$.

So eq. (5) becomes

$$F(D) = \frac{1}{D}\left[D^2\left(1+\frac{\rho a}{3m}\right) + \frac{T}{ma}\left(1+\frac{Ka}{2c}\right)\right],$$

whence

$$y_a = \frac{D}{D^2\left(1+\frac{\rho a}{3m}\right)+\frac{T}{ma}\left(1+\frac{Ka}{2c}\right)} v_0 \ ;$$

taking $\frac{m}{m_0}$ to be equal to unity, we have

$$y_a = \frac{v_0}{c\lambda} \sin \lambda ct \ , \qquad \dots \quad (10)$$

where $\pm i\lambda c$ are the roots of the equation

$$D^2\left(1+\frac{\rho a}{3m}\right)+\frac{T}{ma}\left(1+\frac{Ka}{2c}\right)=0$$

is given by

$$\lambda c = \sqrt{\frac{T}{ma}\left(1+\frac{\rho a}{3m}\right)^{-1}} = \sqrt{\frac{P}{ma} - \frac{P^2}{3m^2}} . \qquad \dots \quad (10.1)$$

Eq. (10) is same as previously obtained in a different paper.

(*iii*) If a be finite, but not small enough to replace coth $\frac{D_k a}{c}$ by the first few terms of the expansion given in eq. (4.2), we have, from eqs. (3) and (4.1),

$$y_a = \frac{1}{D+\frac{T}{mc}\left(1+\frac{K}{2D}\right)\left(\coth\frac{D_k a}{c}+1\right)} . v_0$$

$$= \frac{1-e^{-D_k\theta_1}}{(D+q)-D.e^{-D_k\theta_1}} . v_0 , \qquad \dots \quad (11)$$

where
$$q=\left(\frac{2T}{mc}-\frac{K}{2}\right) = \frac{2pc}{m}\left(1-\frac{Km}{4pc}\right) . \qquad \dots \quad (11.1)$$

On simplification,

$$y_a = \frac{1-e^{-D_k\theta_1}}{D+q}\left[1-\frac{D}{D+q}e^{-D_k\theta_1}\right]^{-1} . v_0$$

$$= \Bigg[\frac{1}{D+q}+\left\{\frac{D}{(D+q)^2}-\frac{1}{D+q}\right\}e^{-D_k\theta_1}+\left\{\frac{D^2}{(D+q)^3}-\frac{D}{(D+q)^2}\right\}e^{-2D_k\theta_1}$$

$$+ + + \left\{\frac{D^n}{(D+q)^{n+1}}-\frac{D^{n-1}}{(D+q)^n}\right\}e^{-nD_k\theta_1}\Bigg] v_0$$

$$= f_1(t)+k[f_2(t-\theta_1)-f_1(t-\theta_1)]+k^2[f_3(t-2\theta_1)-f_2(t-2\theta_1)]$$
$$+ + + k^n[f_{n+1}(t-n\theta_1)-f_n(t-n\theta_1)] , \qquad \dots \quad (11.2)$$

where (*vide* appendix)

$$\frac{1}{D+q} \cdot v_0 = f_1(t) = \frac{v_0}{q}\left(1-e^{-qt}\right)$$

$$\frac{D}{(D+q)^2} \cdot v_0 = f_2(t) = \frac{v_0}{q}e^{-qt} \cdot qt$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\frac{D^n}{(D+q)^{n+1}} \cdot v_0 = f_{n+1}(t) = \frac{v_0}{q}e^{-qt}\Big\{qt - {}^{n-1}C_1\frac{q^2t^2}{2!} + {}^{n-1}C_2\frac{q^3t^3}{3!}$$

$$- \cdots + (-)^{n-1}\,{}^{n-1}C_{n-1}\frac{q^n t^n}{n!}\Big\} \qquad \ldots \quad (11.3)$$

and

$$e^{-D_t n\theta_1} \cdot f_{n+1}(t) = k^n f_{n+1}(t-n\theta_1), \qquad \ldots \quad (11.4)$$

k stands for $e^{-\frac{K}{2}\theta_1}$.

The first term of eq. (11.2) is zero for negative values of t; for positive values, *i.e.*, during $0 < t < \theta_1$,

$$y_a = \frac{v_0}{q}\left(1 - e^{-qt}\right). \qquad \ldots \quad (12)$$

After the time θ_1, the second term no longer vanishes: we have during $\theta_1 < t < 2\theta_1$

$$y_a = y_a(0 < t < \theta_1) - \frac{v_0}{q}k\left[1 - e^{-q(t-\theta_1)} \cdot \Big\{1 + q(t-\theta_1)\Big\}\right]. \qquad \ldots \quad (12.1)$$

Similarly during $2\theta_1 < t < 3\theta_1$,

$$y_a = y_a(\theta_1 < t < 2\theta_1) - \frac{v_0 k^2}{q}e^{-q(t-2\theta_1)} \cdot \frac{q^2(t-\theta_1)^2}{2!}\ ; \qquad \ldots \quad (12.2)$$

during $3\theta_1 < t < 4\theta_1$,

$$y_a = y_a(2\theta_1 < t < 3\theta_1) - \frac{v_0 k^3}{q} \cdot e^{-q(t-3\theta_1)} \cdot \Big\{\frac{q^2(t-3\theta_1)^2}{2!} - \frac{q^3(t-3\theta_1)^3}{3!}\Big\} \qquad \ldots \quad (12.3)$$

and so on.

From eq. (2) the pressure exerted by the hammer, at different intervals, is given by, neglecting the sign for the time being,

$$P_1 = 2\rho v_0 c\, e^{-qt} \qquad (13)$$

$$P_2 = P_1 + 2\rho v_0 c k\, e^{-qt_1}(1 - qt_1) \qquad \ldots \quad (13.1)$$

$$P_3 = P_2 + 2\rho v_0 c k^2\, e^{-qt_2}\left(1 - 2qt_2 + \frac{q^2 t_2^2}{2!}\right) \qquad \ldots \quad (13.2)$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$P_{n+1} = P_n + 2\rho v_0 c k^n\, e^{-qt_n}\left[1 - {}^nC_1 . qt_n + {}^nC_2 . \frac{q^2 t_n^2}{2!} - {}^nC_3 . \frac{q^3 t_n^3}{3!} + - + - (-)^n . {}^nC_n . \frac{q^n t_n^n}{n!}\right] . \quad (13.3)$$

FINITE STRING: HAMMER STRIKES AT THE MID-POINT.

In this case we put $b = a$ in eq. (4) and we get for this case,

$$F(D) = D + q . \coth \frac{D_k a}{c} = D + q\,\frac{1 + e^{-D_k\theta_1}}{1 - e^{-D_k\theta_1}} \qquad \ldots \quad (14)$$

where q is given by eq. (11·1).

The eq. (3) with the help of eq. (15) becomes

$$y_a = \left[\frac{1}{D+q} + \left\{\frac{D-q}{(D+q)^2} - \frac{1}{D+q}\right\} e^{-D_k\theta_1} + \left\{\frac{(D-q)^2}{(D+q)^3} - \frac{(D-q)}{(D+q)^2}\right\} e^{-2D_k\theta_1} + + \ldots + \left\{\frac{(D-q)^n}{(D+q)^{n+1}} - \frac{(D-q)^{n-1}}{(D+q)^n}\right\} e^{-nD_k\theta_1}\right] v_0 \qquad \ldots \quad (15)$$

(*vide* Appendix, *vii*)

$$= f_1(t) + 2k[f_2(t_1) - f_1(t_1)] + 2k^2[2f_3(t_2) - 3f_2(t_2) + f_1(t_2)]$$
$$+ 2k^3[4f_4(t_3) - 8f_3(t_3) + 5f_2(t_3) - f_1(t_1)]$$
$$+ + \ldots + + -$$
$$+ 2k^n\left[2^{n-1} . f_{n+1}(t_n) - 2^{n-2} . \frac{n+1}{n} {}^nC_1 f_n(t_n) + 2^{n-3} \frac{n+2}{n} . {}^nC_2 . f_{n-1}(t_n) - + \ldots - + (-)^n f_1(t_n)\right] . \qquad \ldots \quad (16)$$

The first term is zero for negative values of times; for positive values, *i.e.*, during $0 < t < \theta_1$

$$y_a = f_1(t) = \frac{v_0}{q}\left(1 - e^{-qt}\right). \qquad \dots \quad (16.1)$$

During $\theta_1 < t < 2\theta_1$

$$y_a = f_1(t) + k[2f_2(t_1) - 2f_1(t_1)]$$

$$= y_a(0 < t < \theta_1) - \frac{2v_0 k}{q}\left[1 - e^{-qt_1}(1 + qt_1)\right]. \qquad \dots \quad (16.2)$$

During $2\theta_1 < t < 3\theta_1$

$$y_a = y_a(\theta_1 < t < 2\theta_1) + k^2[4f_3(t_2) - 6f_2(t_2) + 2f_1(t_1)]$$

$$= y_a(\theta_1 < t < 2\theta_1) + \frac{2v_0 k^2}{q}[1 - e^{-qt_2}(1 + qt_2 + q^2t_2^2)] \dots \quad (16.3)$$

During $3\theta_1 < t < 2\theta_1$

$$y_a = y_a(2\theta_1 < t < 3\theta_1) + k^3[8f_4(t_3) - 16f_3(t_3) + 10f_2(t_3) - 2f_1(t_3)]$$

$$= y_a(2\theta_1 < t < 3\theta_1) - \frac{2v_0 k^3}{q}\left[1 - e^{-qt_3}\left(1 + qt_3 + \frac{4}{3!}q^3t_3^3\right)\right] \quad (16.4)$$

and so on.

The pressures exerted by the hammer at different intervals are obtained by the help of (2) and (16), and they are

$$P_1 = 2\rho v_0 c e^{-qt}, \qquad \dots \quad (17.1)$$

$$P_2 = P_1 + 4\rho v_0 c.k e^{-qt_1}(1 - qt_1) \qquad \dots \quad (17.2)$$

$$P_3 = P_2 + 4\rho v_0 c k^2 e^{-qt_2}(1 - 3qt_2 + q^2t_2^2) \qquad \dots \quad (17.3)$$

$$P_4 = P_3 + 4\rho v_0 c k^3 e^{-qt_3}\left(1 - 5qt_3 + 4q^2t_3^2 - \frac{4}{3!}q^4t_3^4\right) \qquad \dots \quad (17.4)$$

$$P_5 = P_4 + 4\rho v_0 c k^4 e^{-qt_4}\left(1 - 7qt_4 + q^2t_4^2 - \frac{20}{3!}q^3t_4^3 + \frac{8}{4!}q^4t_4^4\right),$$

$$\dots \quad (17.5)$$

and so on.

When the hammer strikes very near the end of a long string, we find from eq. (9) that, at $t = 0$ pressure increases by sudden jump, in magnitude $2\mu v_0 m$ which is equal to $\rho v_0 c$ for an undamped string. Then it continuously rises, attains a maximum value and finally falls slowly to zero at the end of the impact, the duration of which is given by $P = 0$. On the other hand, considering the rigorous expressions for pressure at different intervals as given by eqs (13), (13.1) etc., we find that so long as the hammer is in contact, the pressure fluctuates in a definite discontinuous manner. The pressure increases by sudden jump in magnitude $2\rho v_0 c$ at t equal to zero, whence it falls down slowly to a minimum value till at $t = \theta_1$ the pressure again rises suddenly in magnitude $2\rho v_0 c k$. This process continues, with the sudden rise of pressures in magnitudes $2\rho v_0 c$, $2\rho v_0 ck$, $2\rho v_0 ck^2$, $2\rho v_0 ck^3$ at $t = 0$, θ_1, $2\theta_1$, $3\theta_1$ respectively, till the impact terminates. The presence of k whose magnitude is less than unity lowers the successive magnitudes of the sudden rises of pressure occurring at the beginning of each interval. If the string is free from damping the pressure evidently rises by sudden jumps of constant magnitude $2\rho v_0 c$. The nature of the pressure

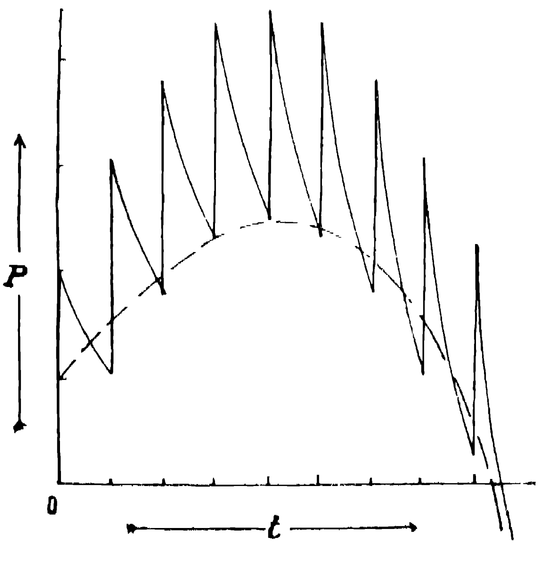

FIGURE 1.

fluctuation, as following from the theory, is shown graphically in Figure 1. The continuous curve from eq. (9) is shown doted in the figure. The discontinuous curve represents the pressure as given by eqs (13), (13.1) etc. for undamped string. It is evident from the graph that the continuous curve represents approximately the mean values of the discontinuous curve; and it will be the actual mean curve when the number of discontinuities is very large, which is the case when the length of shorter segment is very small.

The pressure exerted by the hammer when it strikes at the mid-point of a finite string also fluctuates discontinuously, as is evident from eqs. (18), (18·1) etc. But the sudden rise of pressure at $t = 0$ is $2\rho v_0 c$, but subsequently at $t = \theta_1, 2\theta_1 \ldots\ldots$, the rises are constant and of magnitude $4\rho v_0 c$ in the case of undamped string, till the pressure terminates.

My best thanks are due to Prof. K. C. Kar, D.Sc., of Presidency College, Calcutta, who has taken a lively interest in this work.

APPENDIX.

(*i*)
$$\coth x = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}} = \frac{1 + e^{-2x}}{1 - e^{-2x}} \qquad \ldots \quad (1)$$

$$= \frac{1}{x}\left\{1 + B_1 \frac{(2x)^2}{2\,!} - B_3 \frac{(2x)^4}{4\,!} + B_5 \frac{(2x)^6}{6\,!} - + \ldots\right\}, \qquad (2)$$

where $B_1 = \frac{1}{6}$, $B_3 = \frac{1}{30}$, $B_5 = \frac{1}{42}$, $B_7 = \frac{1}{30}$, $B_9 = \frac{5}{66}$ and they are known as Bernoullian numbers.

(*ii*)
$$Lt.\ x \to 0 \quad \coth x \to \frac{1}{x}\left(1 + \frac{x^2}{3} - \frac{x^3}{45}\right) \to O(\infty) \qquad \ldots \quad (3)$$

$$Lt\ x \to \infty \quad \coth x \to 1, \text{ from eq. (1)} \qquad \ldots \quad (4)$$

(*iii*)
$$\frac{D}{(D+q)^n} \cdot v_0 = D^{-(n-1)} \cdot (1 + qD^{-1}).\ v_0$$

$$= \left[D^{-(n-1)} - nqD^{-n} + \frac{n(n+1)}{2\,!} q^2 D^{-(n+1)} - \frac{n(n+1)(n+2)}{3\,!} q^3 D^{-(n+2)} + - \ldots\right] v_0.$$

$$= v_0 \left[\frac{t^{n-1}}{(n-1)\,!} - nq\frac{t^n}{n\,!} + n(n+1)\, q^2 \frac{t^{n+1}}{(n+1)\,!} - + \ldots\right]$$

$$= v_0 \frac{t^{n-1}}{(n-1)\,!} e^{-qt} \qquad \ldots \quad (5)$$

(*iv*)
$$\frac{D^n}{(D+q)^{n+1}} \cdot v_0 = D^{n-1} \cdot \frac{D}{(D+q)^{n+1}} \cdot v_0$$

$$= v_0\, D^{n-1} \frac{t^n}{n\,!} e^{-qt}$$

$$= v_0 e^{-qt} (D - q)^{n+1} \frac{t^n}{n!}$$

$$= v_0 e^{-qt} \left[D^{n-1} - {}^{n-1}C_1 \cdot D.^{n-2}q + {}^{n-2}C_2 \cdot D.^{n-3}q^2 + - \right.$$

$$\left. + - + (-)^{n-1}q^{n-1} \right] \frac{t^n}{n!}$$

$$= \frac{v_0}{q} e^{-qt} \left[qt - {}^{n-1}C_1 \frac{q^2t^2}{2!} + {}^{n-1}C_2 \frac{q^3t^3}{3!} \right.$$

$$\left. - + \dots + (-)^{n-1} \frac{q^nt^n}{n!} \right]$$

$$= f_{n+1}(t) ; \qquad (6)$$

whence we have from eq. (5) or (6)

$$\frac{1}{D+q} v_0 = \left[1 - \frac{D}{D+q} \right] \frac{v_0}{q} = \frac{v_0}{q} (1 - e^{-qt}) = f_1(t) \qquad (7)$$

and
$$\frac{D}{(D+q)^2} v_0 = f_2(t) = v_0 \frac{e^{-qt}}{q} . qt \qquad (8)$$

$$\frac{D^2}{(D+q)^3} v_0 = f_3(t) = v_0 \frac{e^{-qt}}{q} (qt - \frac{q^2t^2}{2!}); \qquad (9)$$

$$\frac{D^3}{(D+q)^4} v_0 = f_4(t) = v_0 \frac{e^{-qt}}{q} (qt - 2\frac{q^2t^2}{2!} + \frac{q^3t^3}{3!}); \qquad (10)$$

and so on.

(*v*) $$e^{-\lambda D} . f_n(t) = (1 - \lambda D + \frac{\lambda^2 D^2}{2!} - \frac{\lambda^3 D^3}{3!} + .) f_n(t)$$

$$= \left\{ f_n(t) - \lambda f'(t) + \frac{\lambda^2}{3!} f''(t) - \frac{\lambda^3}{2!} f'''(t) + \dots \right\}$$

$$= f_n(t - \lambda) \qquad (11)$$

(*vi*) $$\frac{(D-q)^n}{(D+q)^{n+1}} . v_0 = \frac{(2D - D_1)^n}{D_1^{n+1}} . v_0,$$ where D_1 stands for $(D+q)$

$$= [2^n D^n - {}^nC_1 2^{n-1} . D^{n-1} . D_1 + {}^nC_2 2^{n-2} . D^{n-2} D_1^2$$

$$- + \dots \dots + (-)^n D_1^n] D_1^{-(n+1)} . v_0$$

$$[2^n \frac{D^n}{(D+q)^{n+1}} - {}^nC_1 . 2^{n-1} \frac{D^{n-1}}{(D+q)^n} + {}^nC_2 2^{n-2} \frac{D^{n-2}}{(D+q)^n} ,$$

$$- + \dots \dots + (-)^n \cdot \frac{1}{D+q} \Big] v_0$$

$$= \left[2^n f_{n+1}(t) - {}^nC_1 . 2^{n-1} f_n(t) + {}^nC_2 2^{n-2} f_{n-1}(t) \right.$$

$$\left. - + \dots + (-)^n f_1(t) \right]. \qquad \dots \quad (12)$$

(vii) $$\left[\frac{(D-q)^n}{(D+q)^{n+1}} - \frac{(D-q)^{n-1}}{(D+q)^n} \right] v_0, \text{ (from eq. (12))}$$

$$= 2^n f_{n+1}(t) - 2^{n-1} \cdot \frac{n+1}{n} \cdot {}^nC_1 f_n(t) + 2^{n-2} \cdot \frac{n+2}{n} \, {}^nC_2 f_{n-1}(t)$$

$$- 2^{n-3} \cdot \frac{n+3}{n} \, {}^nC_3 f_{n-2}(t) + - \dots + (-)^n 2 f_1(t), \quad \dots \quad (13)$$

whence,

$$\left[\frac{D-q}{(D+q)^2} - \frac{1}{D+q} \right] v_0 = 2f_2(t) - 2f_1(t) \quad \text{from eq. (12) and (7);} \quad \dots \quad (14)$$

$$\left[\frac{(D-q)^2}{(D+q)^3} - \frac{D-q}{(D+q)^2} \right] v_0 = 4f_3(t) - 6f_2(t) + 2f_1(t). \quad \text{from eq. (13);} \quad \dots \quad (15)$$

$$\left[\frac{(D-q)^3}{(D+q)^4} - \frac{(D-q)^2}{(D+q)^3} \right] v_0 = 8f_4(t) - 16f_3(t) + 10f_2(t) - 2f_1(t); \quad \dots \quad (16)$$

and so on.

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